Fluctuations in first passage percolation

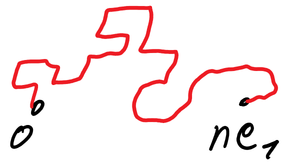
Thursday, 13 May 2021 13:09

On $\mathbb{Z}^d$, assign to every edge e, a random weight η_e with (η_e) IID with common dist. ν. e.g. exponential, Bernoulli(p), uniform.
The weight of a path p in $\mathbb{Z}^d$ is defined as the sum of the η_e along the edges of p.
Get a random metric space with

$$T(u,v) = \inf_{\substack{\text{paths } p \\ \text{from } u \text{ to } v}} \text{weight of } p, \quad u,v\in\mathbb{Z}^d.$$

Predictions: We saw that $T(0,nx) = n(\mu(x)+o(1))$ (some number depending on x, the time constant) for each $x\in\mathbb{Q}^d$, as $n\to\infty$, a.s. and in L^1 (under suitable assump. on ν). precise meaning not clear

Time fluctuat. $\exists\, \chi(d)\geq 0$ s.t. $\mathrm{Var}(T(0,nx))\approx n^{2\chi(d)}$
(n^χ is the std. dev. of the passage time)
Exponent χ should be universal, the same for all ν under mild assumptions, and the same for all x. Obviously, in $d=1$, $\chi=\frac{1}{2}$.
In $d=2$, predicted $\chi=\frac{1}{3}$.
predicted that $\chi(d)\leq\chi(d-1)$.
Disputed whether $\chi(d)=0$ for d large.



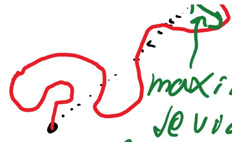
$e_1=(1,0,\ldots,0)$.

In $d=2$, $\dfrac{T(0,ne_1)-n\mu(e_1)}{n^{1/3}} \xrightarrow[n\to\infty]{d}$ Tracy-Widom F_2 dist. (fluctuations of largest eigenvalue of a GUE random matrix)
(Hermitian complex indep. Gaussian entries, Gaussian Unitary Ensemble)

For "solvable models" (all of them are directed last passage percolation) this has been shown. Moreover, a scaling limit "The directed landscape" was constructed in 2018 by Dauvergne-Ortmann-Virág.

Transversal fluctuations: By how much do geodesics (weight minimizing paths) deviate from the straight line? in $d\geq 2$

By how much do geodesics deviate from the straight line? It is predicted that in $d \geq 2$ there is a $\xi(d) \geq \frac{1}{2}$ s.t. the deviation $\approx n^{\xi}$.



maximal deviation from straight line.

$\underbrace{\xi(d) \geq \tfrac{1}{2}}$ deviate more than simple random walk

In $d=2$, predicted $\xi = \frac{2}{3}$.

predicted relation: In every dimension $\chi(d) = 2\xi(d) - 1$.

Known (for non-solvable models. E.g., First-passage percolation with $\nu(\{0\}) < p_c(d)$ and having exponential moments):

$$c \leq Var(T(0, ne_1)) \leq \frac{C n}{\log n} \quad \text{for every } d \geq 2.$$

Benjamini-Kalai-Schramm (2003) and follow-up

In $d=2$: $Var(T(0, ne_1)) \geq c \log n$. Newman-Piza (1995)

(e.g. $0 \leq \chi \leq 1/2$).

Licea-Newman-Piza (1996): Also, $\xi \geq \frac{1}{d+1}$ for every $d \geq 2$. For stronger def. of ξ, $\xi \geq \frac{1}{2}$ for every $d \geq 2$ and $\xi \geq \frac{3}{5}$ in $d=2$.

open: Transversal fluc. are $o(n)$.

There also exist conditional results. E.g., conditioned on the limit shape being uniformly (strictly) convex.

Reference: Auffinger-Damron-Hanson 50 years of FPP.

To explain the proofs of some of these results we take some detours.

Concentration inequalities - Efron-Stein ineq.

Let $X_1, \dots, X_n$ be RVs taking values in a meas. space $\mathcal{X}$. Let $f: \mathcal{X}^n \to \mathbb{R}$ be s.t. $\mathbb{E}(f^2(X)) < \infty$. Would like to bound $Var(f)$ from above.

$X = (X_1, \dots, X_n)$.

σ-alg. $= \{\emptyset, \text{Full}\}$

Would like to bound variation above. $X = (X_1, \dots, X_n)$.

Let $\mathcal{F}_0 :=$ trivial σ-alg. $= \{\emptyset, \text{full set}\}$

$\mathcal{F}_i := \sigma(X_1, \dots, X_i)$ for $1 \le i \le n$.

Doob Martingale: $M_i := \mathbb{E}(f(X) \mid \mathcal{F}_i)$

Martingale difference: $\Delta_i := M_i - M_{i-1}$.

General Variance formula: $\mathrm{Var}[f(X)] = \sum_{i=1}^n \mathbb{E}(\Delta_i^2)$

Proof: $f(X) - \mathbb{E} f(X) = \sum_{i=1}^n \Delta_i$.

Note $\mathbb{E}\Delta_i = 0$ by def., for each $1 \le i \le n$.

Additionally, $\mathbb{E}\Delta_i \Delta_j = 0$ for $i \ne j$.

Since, for $i > j$, $\mathbb{E}\Delta_i \Delta_j = \mathbb{E}[\mathbb{E}(\Delta_i \Delta_j \mid \mathcal{F}_j)] =$

$= \mathbb{E}[\Delta_j \underbrace{\mathbb{E}(\Delta_i \mid \mathcal{F}_j)}_{=0}]$

$= 0$, by mart. prop. of M_i, $\mathbb{E}(M_m \mid \mathcal{F}_j) = M_j$ when $m \ge j$.

$\Rightarrow \mathrm{var}(f(X)) = \mathbb{E}((f(X) - \mathbb{E} f(X))^2) =$

$= \mathbb{E}((\sum_{i=1}^n \Delta_i)^2) = \sum_{i=1}^n \mathbb{E}(\Delta_i^2)$.

To proceed, we will assume that the (X_i) are independent.

Define the σ-alg., $G_i = \sigma(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$.

One notion of influence: $I_i := \mathbb{E}[\mathrm{Var}(f(X) \mid G_i)] =$

$= \mathbb{E}[(f(X) - \mathbb{E} f(X \mid G_i))^2]$

Equivalent expressions: If $X^i = (X_1, \dots, X_{i-1}, X_i', X_{i+1}, \dots, X_n)$ with X_i' indep. and dst. as X_i.

Then $I_i = \frac{1}{2}\mathbb{E}[(f(X) - f(X^i))^2] = \mathbb{E}[(f(X) - f(X^i))_+^2] =$

$= \mathbb{E}[(f(X) - f(X^i))_-^2]$

(If Z, Z' are IID real-valued then

$\mathbb{E}[(Z - \mathbb{E}Z)^2] = \frac{1}{2}\mathbb{E}[(Z - Z')^2] = \mathbb{E}((Z - Z')_+^2) = \mathbb{E}((Z - Z')_-^2)$

$(Z - Z')^2 = (Z - Z')_+^2 + (Z - Z')_-^2$ IID

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Lastly, $I_i = \inf_{\substack{Z_i \text{ RV} \\ \text{meas. wrt. } G_i}} \mathbb{E}\left[(f(X)-Z_i)^2\right].$

(If Z is real-valued then $\operatorname{var}(Z) = \inf_{a\in\mathbb{R}} \mathbb{E}\left[(Z-a)^2\right]$)

Thm. (Efron-Stein 1981, Steele 1986, Rhee-Talagrand 1986): When $X_1, \dots, X_n$ are indep., $\operatorname{var}(f(X)) \le \sum_{i=1}^n I_i$.

Proof: $\operatorname{var}(f) = \sum_{i=1}^n \mathbb{E}(\Delta_i^2) = \sum_{i=1}^n \mathbb{E}\Big[\big(\underbrace{\mathbb{E}f(X)|\mathcal{F}_i) - \mathbb{E}f(X)|\mathcal{F}_{i-1})}\big)^2\Big]$

independence $\Rightarrow = \mathbb{E}[f(X) - \mathbb{E}(f(X)|G_i) | \mathcal{F}_i]$

$$= \sum_{i=1}^n \mathbb{E}\left[\left(\mathbb{E}\left[f(X) - \mathbb{E}(f(X)|G_i) \,|\, \mathcal{F}_i\right]\right)^2\right] \le \leftarrow \text{For real-valued } Z \;\; (\mathbb{E}Z)^2 \le \mathbb{E}(Z^2)$$

$$\le \sum_{i=1}^n \mathbb{E}\left[\left(f(X) - \mathbb{E}(f(X)|G_i)\right)^2\right] = \sum_{i=1}^n I_i .$$

Corollary (bounded differences inequality):

Let $X_1, \dots, X_n$ be indep. taking values in $\mathcal{X}$.
Let $f: \mathcal{X}^n \to \mathbb{R}$ be meas., $\mathbb{E}(f^2(X)) < \infty$ and
$\forall\, 1 \le i \le n$, $|f(x) - f(x')| \le c_i$ when x, x' differ only in i'th coord.

Then $\operatorname{var}(f(X)) \le \frac{1}{4}\sum_{i=1}^n c_i^2$.

Proof: By Efron-Stein,

$\operatorname{var}(f(X)) \le \sum_{i=1}^n I_i$

It is easy to see that $I_i \le c_i^2$. To get the $\frac{1}{4}$,
write $I_i = \inf_{\substack{Z_i \text{ RV} \\ \text{meas. wrt. } G_i}} \mathbb{E}\left[(f(X)-Z_i)^2\right] \le \mathbb{E}\left[\left(\frac{1}{2}c_i\right)^2\right] \le \frac{1}{4}c_i^2.$

$Z_i = \frac{1}{2}\big(\sup f(X) \text{ given } G_i - \underbrace{\inf f(X) \text{ given } G_i}\big)$
the diff. is at most c_i

Application (Longest Common subseq.):

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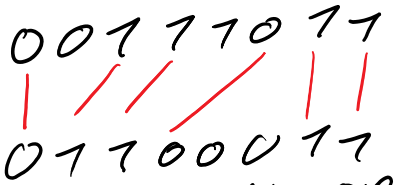
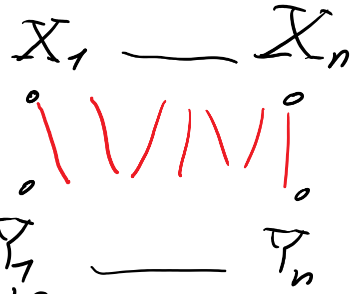
$X_1, \dots, X_n, Y_1, \dots, Y_n$ indep. unif. on $\{0,1\}$.





Length of the

$f(X,Y) =$ longest common subseq.

We saw $\frac{f(X,Y)}{n} \xrightarrow[n\to\infty]{} c$ a.s. and in L^1.

close to 80%.

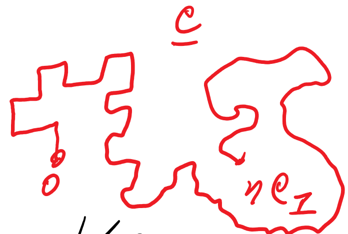
Here, we note $Var(f(X,Y)) \le \frac{1}{2} n$.

This is an immed. applic. of the bdd. diff. ineq. with $c_i = 1$ for all i.

Open: $Var(f(X,Y)) \xrightarrow[n\to\infty]{} \infty$.

Cons.: $Var(f(X,Y)) = \Theta(n)$.

Application (first passage percolation)



$\exists\, C = C(\nu)$, $Var(T(0, ne_1)) \le C \cdot n$, $\forall n$.

This is known for a very general class of weight dist. ν. We'll show it now when the weight takes values in $[a,b]$, $0<a\le b<\infty$.

Proof: Since the weight is bdd., the geodesic necessarily stays in a ball of radius Cn (the length of the geodesic is $\le \frac{b}{a} \cdot n$) around 0, so $T(0,ne_1)$ is a fcn. of finitely many (indep.) edge weights.

Thus $Var(T(0,ne_1)) \le \sum_e I_e$ ← Efron-Stein ineq.

for edges e in the ball.

$I_e = \mathbb{E}[Var(T(0,ne_1)) /$ cond. on all ...$]$.

for edge e

$$I_e = \mathbb{E}\left[\mathrm{Var}\left(T(0,ne_1) \,\middle/\, \begin{matrix}\text{cond. on all}\\ \text{but } \eta_e\end{matrix}\right)\right].$$

Let η^e be another set of weights with $\eta^e_f = \eta_f$ for all edges f, except $f=e$

where η^e_e is an indep. copy of η_e.

Using an equiv. repr. of I_e,

$$I_e = \mathbb{E}\Big[\underbrace{\left(T(0,ne_1)(\eta^e) - T(0,ne_1)(\eta)\right)_+^2}_{\le \mathbb{1}_{e \text{ is on all geodesics from } 0 \text{ to } ne_1}(\eta)\cdot(\eta^e_e)^2}\Big] =$$

passage time can only increase if all geodesics from 0 to ne_1 pass through e.

indep. random variables

$$= \mathbb{P}\left(\begin{matrix}e \text{ is on}\\ \text{all geodesics}\end{matrix}\right)\cdot \underbrace{\mathbb{E}\left((\eta^e_e)^2\right)}_{\le C_1}$$

$$\Rightarrow \mathrm{Var}(T(0,ne_1)) \le C_1 \cdot \mathbb{E}\Bigg(\underbrace{\begin{matrix}\text{length of the}\\ \text{geodesic from}\\ 0 \text{ to } ne_1\end{matrix}}_{\text{as mentioned}, \ \le \frac{b}{a}n}\Bigg) \le C(\nu)\cdot n.$$

Can put $\mathbb{E}\left(\begin{matrix}\text{number of}\\ \text{edges common}\\ \text{to all geodesics}\end{matrix}\right)$

Improved upper bound on the variance

To improve the Efron-Stein upper bound on $T(0,ne_1)$ we will discuss "improved influence ineq.". These go back to Talagrand (1994) and were extended, e.g., by Rossignol (2006) and Falik-Samorodnitsky (2007). We'll use the latter ineq.

the latter ineq.

Let $X_1, \ldots, X_n$ be indep. Let f be a real-valued fcn. of X with suitable moment assump.

Define $Ent(f(X)) := \mathbb{E}[f \log f] - \mathbb{E}(f)\log(\mathbb{E}f)$

(we shorthand $f(X)$ to f).

Thm. (Falik-Samorodnitsky):

$$Var(f) \cdot \log\left[\frac{Var(f)}{\sum_{k=1}^{n} (\mathbb{E}|\Delta_k|)^2}\right] \leq \sum_{k=1}^{n} Ent(\Delta_k^2)$$

where $\Delta_k = \mathbb{E}(f(X)|\mathcal{F}_k) - \mathbb{E}(f(X)|\mathcal{F}_{k-1})$

and $\mathcal{F}_k = \sigma(X_1, \ldots, X_k)$

(recall $Var f = \sum_{k=1}^{n} \mathbb{E}(\Delta_k^2)$).